Substitutions for Planar n-Fold Tilings

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Theorem

$\forall n \in \mathbb{N}$, there exists a planar substitution tiling with n-fold symmetry

Context:

n-fold tiling: any finite pattern also appears up to a $\frac{2\pi}{n}$ rotation (local symmetry)

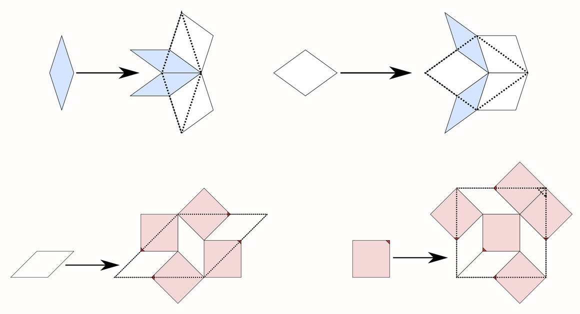
quasiperiodic tiling: any finite pattern appears within bounded distance from any point

planar tiling: obtained by a cut-and-project scheme with physical space $\mathcal{E}$ and internal space $\mathcal{E}^{\perp}$, it has sharp diffraction peaks

substitution tiling: invariant under a replacement of each tile by a (more or less) homothetic pattern : it has a hierarchical structure

Related results:

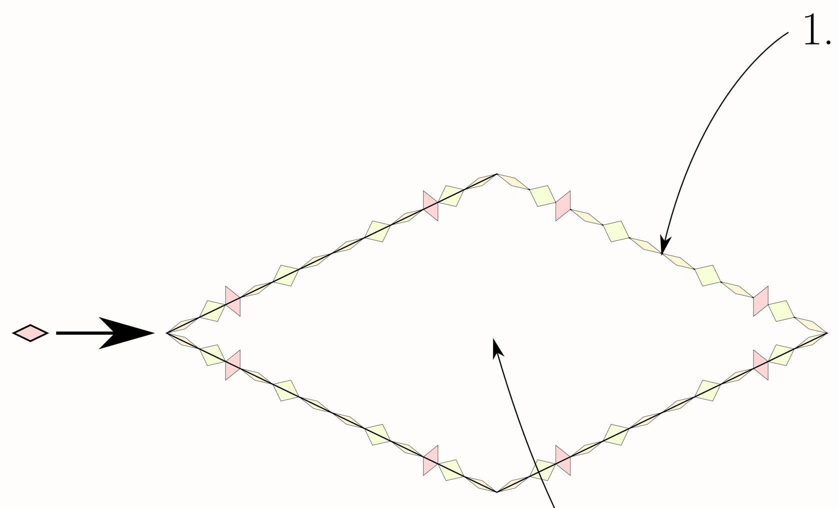
- Ammann-Beenker and Penrose provide classical solutions for $n = 4$ and $n = 5$



- SubRosa tilings provide non-planar substitutions for n-fold tilings
  - J. Kari, M.Rissanen, *Sub Rosa, A System of Quasiperiodic Rhombic Substitution Tilings with n-Fold Rotational Symmetry*, Discrete and Computational Geometry (June 2016).

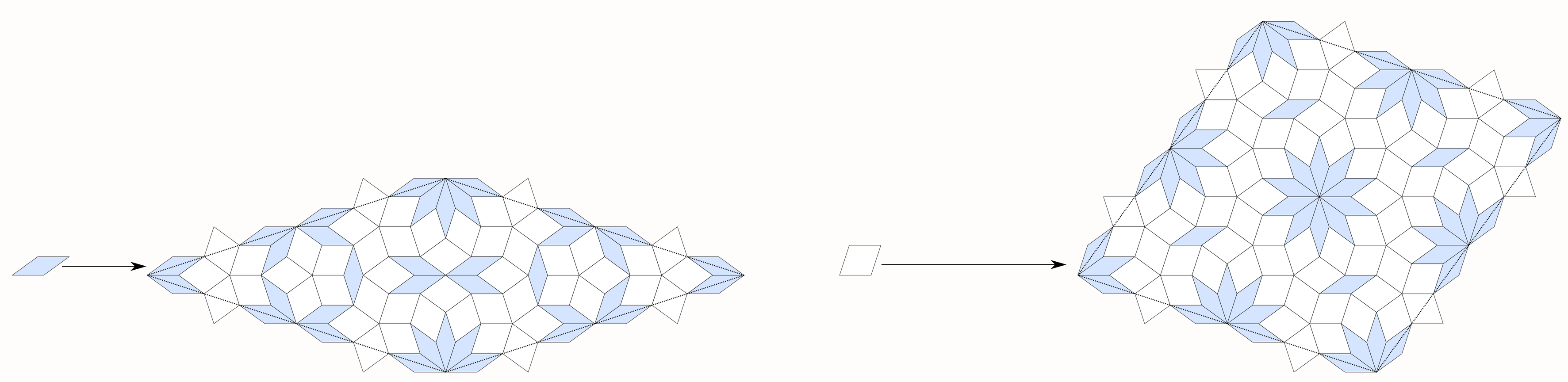
Sketch of the Proof:

To get a substitution for a planar n-fold tiling there are two difficulties:



1. The boundary of the substitution defines the *expansion*, which when we lift the tiling to $\mathbb{Z}^n$ defines a linear application φ. For the substitution to be planar we need φ to be dilating along $\mathcal{E}$ and contracting along $\mathcal{E}^{\perp}$. So we start by defining only the expansion, *i.e.* define the substitution only on the boundary, with constraints that give us the eigenspaces and eigenvalues we want.
2. Once the boundary is defined we need to fill the interior, as long as the interior is tilable it does not affect the planarity.

Example 5-fold:



The expansion of this substitution is $\varphi = \begin{pmatrix} 4 & 0 & -4 & -2 & 2 \\ 2 & 4 & 0 & -4 & -2 \\ -2 & 2 & 4 & 0 & -4 \\ -4 & -2 & 2 & 4 & 0 \\ 0 & -4 & -2 & 2 & 4 \end{pmatrix}$

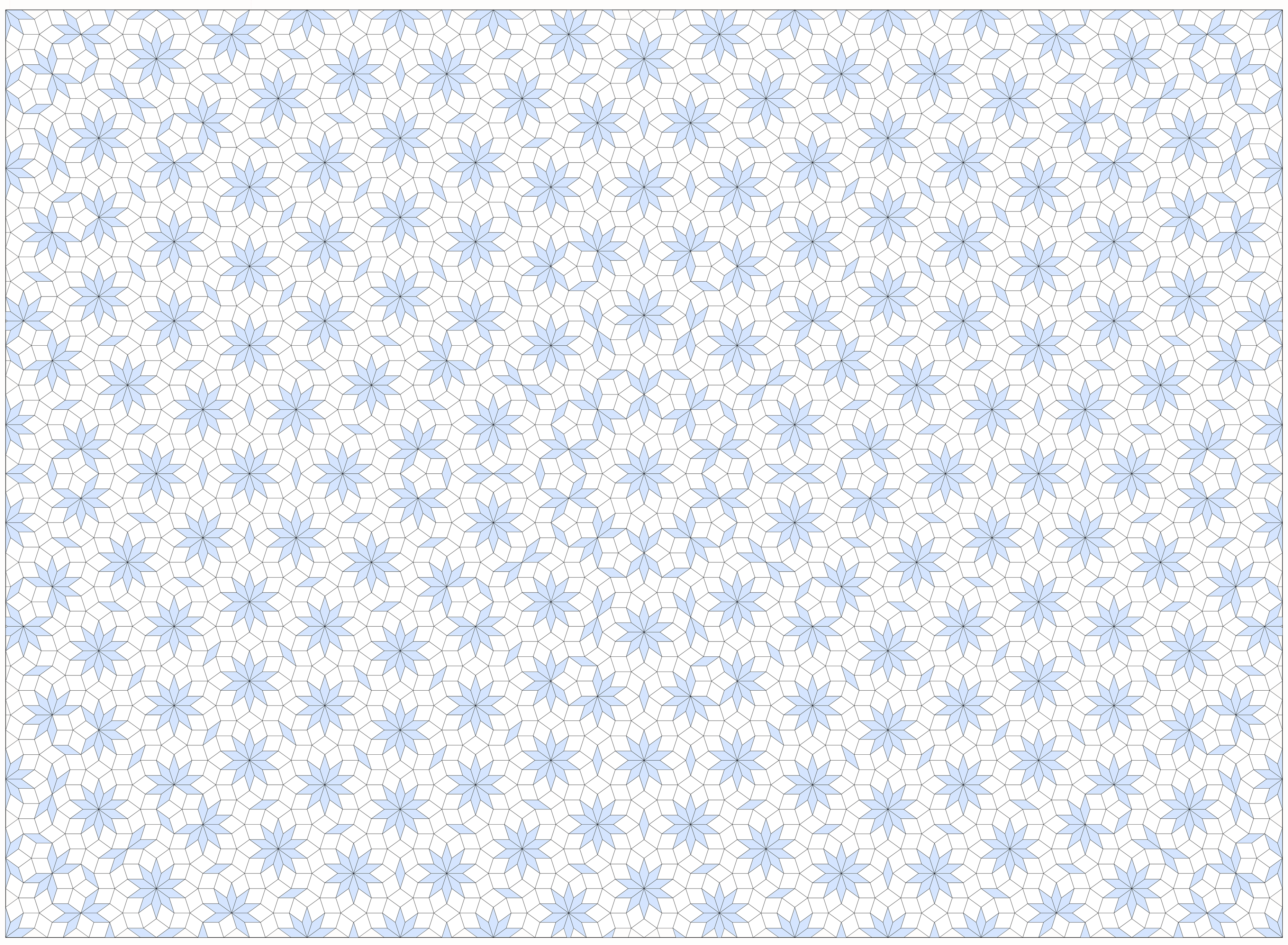
with eigenspaces and eigenvalues:

$$\mathcal{E}_1 = vect\left(\left(\cos(\frac{2i\pi}{5})\right)_{0\leq i<4}, \left(\sin(\frac{2i\pi}{5})\right)_{0\leq i<4}\right) = \text{ Penrose plane}$$

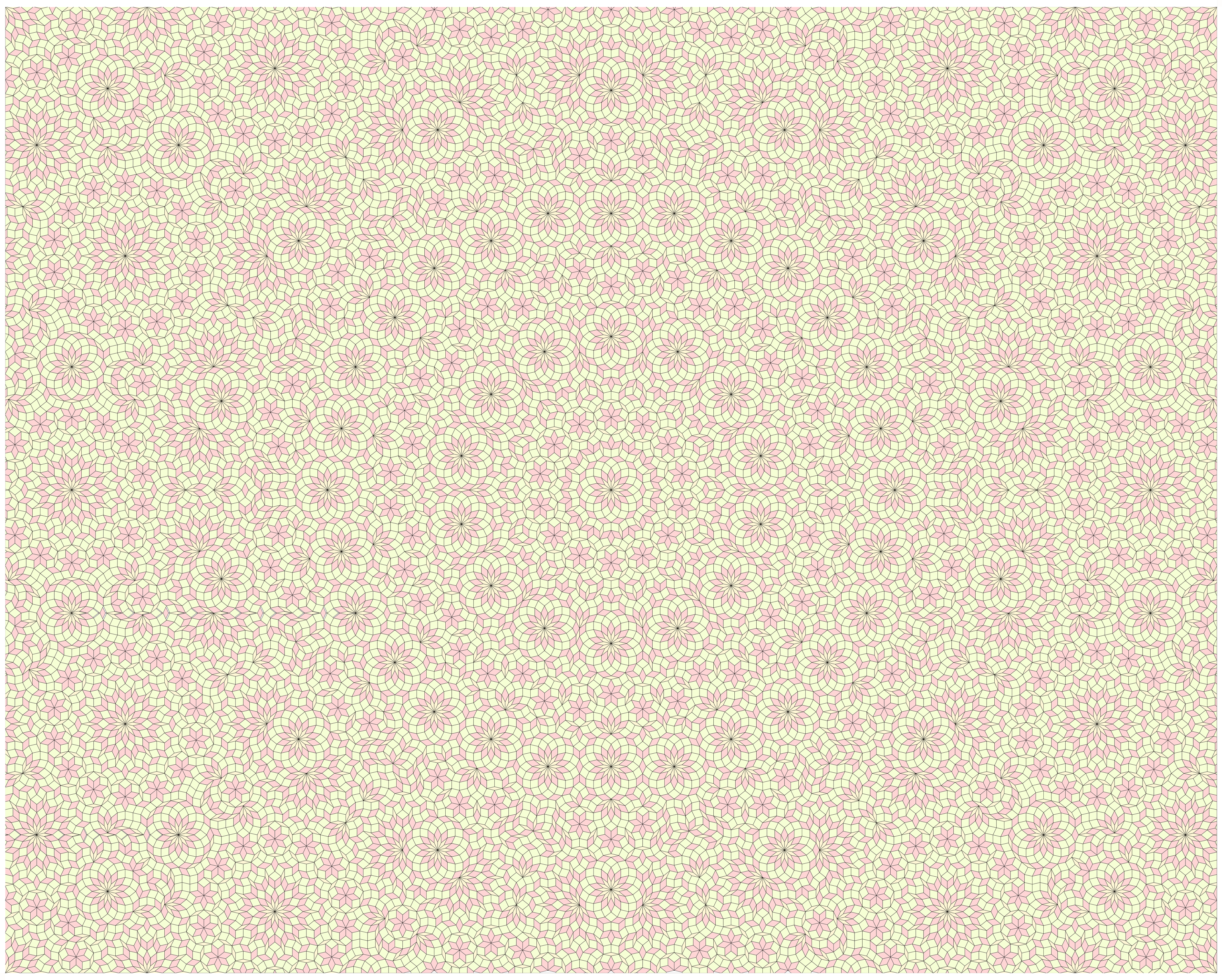
$$\mathcal{E}_2 = vect\left(\left(\cos(\frac{4i\pi}{5})\right)_{0\leq i<4}, \left(\sin(\frac{4i\pi}{5})\right)_{0\leq i<4}\right)$$

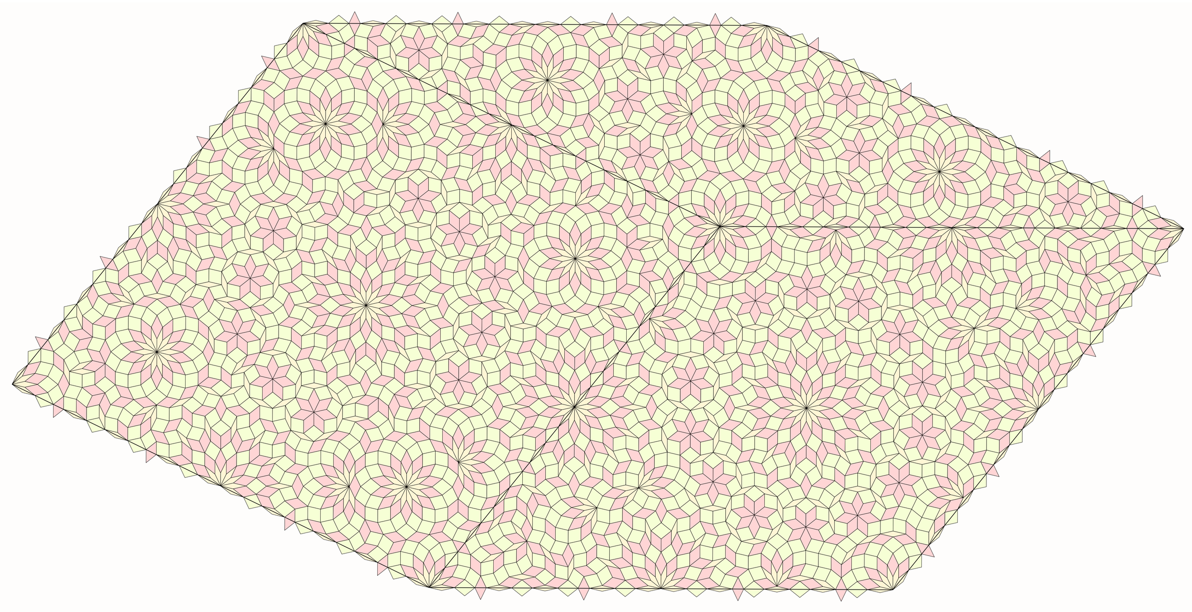
$$\Delta = vect((1,1,1,1,1)) \qquad \lambda_\Delta = 0$$

$$\lambda_1 = 8\cos\left(\frac{\pi}{10}\right) + 4\cos\left(\frac{3\pi}{10}\right) \approx 9.96 \qquad \lambda_2 = 8\cos\left(\frac{3\pi}{10}\right) + 4\cos\left(\frac{9\pi}{10}\right) \approx 0.90$$



Example 7-fold:





$$\varphi = \begin{pmatrix} 10 & 4 & -4 & -10 & -8 & 0 & 8 \\ 8 & 10 & 4 & -4 & -10 & -8 & 0 \\ 0 & 8 & 10 & 4 & -4 & -10 & -8 \\ -8 & 0 & 8 & 10 & 4 & -4 & -10 \\ -10 & -8 & 0 & 8 & 10 & 4 & -4 \\ -4 & -10 & -8 & 0 & 8 & 10 & 4 \\ 4 & -4 & -10 & -8 & 0 & 8 & 10 \end{pmatrix}$$

$$\lambda_1 = 20\cos\left(\frac{\pi}{14}\right) + 16\cos\left(\frac{3\pi}{14}\right) + 8\cos\left(\frac{5\pi}{14}\right) \approx 35.49$$

$$\lambda_2 \approx 0.90,\ \lambda_3 \approx 0.67,\ \lambda_\Delta = 0$$